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Finite-gap difference operators with elliptic coefficients and their spectral curves

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Abstract

We review recent results on the finite-gap properties of difference operators with elliptic coefficients and give explicit characterization of spectral curves for difference analogues of the higher Lamé operators. This curve parametrizes double-Bloch solutions to the difference Lamé equation. The curve depends on a positive integer number ℓ, related to its genus g by $g = 2\ell$, and two continuous parameters: the lattice spacing η and the modular parameter τ. Isospectral deformations of the difference Lamé operator under Volterra flows are also discussed.

1 Introduction

The spectrum of the Schrödinger operator $-\partial_x^2 + u(x)$ with a periodic potential $u(x) = u(x+T)$ has a band structure: there are stable energy bands separated by gaps. For smooth potentials, the width of gaps rapidly decreases as energy becomes higher. However, gaps generically occur at arbitrarily high energies, so there are infinitely many of them.

Of particular interest are exceptional cases, when for sufficiently high energies there are no gaps anymore and their number is therefore finite. Such operators are usually refered to as algebraically integrable or finite-gap ones. Their studies go back to classical works of the last century. The renewed interest to the theory of finite-gap operators is due to their role in constructing quasi-periodic exact solutions to non-linear integrable equations.

Among examples of the finite-gap operators, the first and the most familiar one is the classical Lamé operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + \ell(\ell+1)\wp(x+\omega'|\omega,\omega') \,, \tag{1.1}$$

where $\wp(x|\omega,\omega')$ is the Weierstrass $\wp$-function and ℓ is a parameter. The potential is a doubly-periodic function on the complex plane with periods 2ω and $2\omega'$, where $\mathrm{Im}\,(\omega'/\omega) = \tau > 0$. If ω is real while ω' is pure imaginary, the spectral problem is self-adjoint. The

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finite gap property of higher Lamé operators for integer values of ℓ was established in [1]. If ℓ is a positive integer, then the Lamé operator has exactly ℓ gaps in the spectrum. Such a remarkable spectral property is a signification of a hidden algebraic symmetry which, in its turn, leads to an intimate connection with integrable systems.

The finite-gap property becomes even more striking for difference operators. A natural difference analogue of the Schrödinger equation has the form

$$a(x)\Psi(x+\eta)+b(x)\Psi(x)+c(x)\Psi(x-\eta)=E\Psi(x)\,, \tag{1.2}$$

where the parameter η is the lattice spacing. Let us assume that η is real, the coefficients are real functions of x and $c(x)=a(x-\eta)$, then the problem is self-adjoint. Let the coefficient functions be periodic with a common period T: $a(x+T)=a(x)$, $b(x+T)=b(x)$. The difference Schrödinger operators with periodic coefficients exhibit much richer spectral properties because the problem has two competing periods (T and η) rather then one. Nevertheless, the class of finite-gap operators survives.

The structure of the spectrum of a typical difference operator crucially depends on whether the ratio T/η of the two periods is rational or irrational number. In the former (commensurate) case one can always set $T/\eta=Q\in\mathbf{Z}$ without loss of generality. Then there are no more than Q stable bands in the spectrum. Indeed, set $\Psi(x_0+n\eta)=\Psi_n$, $a(x_0+n\eta)=a_n$, etc and rewrite eq. (1.2) in the form

$$a_n\Psi_{n+1}+b_n\Psi_n+c_n\Psi_{n-1}=E\Psi_{n,}\,,$$

where $a_{n+Q}=a_n$, $b_{n+Q}=b_n$, $c_{n+Q}=c_n$. Since the coefficients are periodic, one may look for solutions in the Bloch form: $\Psi_n=e^{ik\eta n}\chi_n$, where χ_n is Q-periodic and k is the Bloch momentum. Therefore, the spectral problem is reduced to the eigenvalue problem for a hermitian $Q\times Q$-matrix. For each real value of the Bloch momentum k the secular equation has Q real solutions $E=E_i(k)$. As k sweeps over the Brillouin zone, $E_i(k)$ sweep over the stable bands labeled by i. Several neighbouring bands can merge, so the total number of stable bands can be less or equal to Q.

The latter, incommensurate case can be practically realized as a proper limit of the former when both numerator and denominator of the fraction $T/\eta=Q/P$ tend to infinity. The resulting spectra can be (and usually are) extremely complicated chaotic generations of Cantor set type. Some of them, like those in the Azbel-Hofstadter problem [2], though of a multifractal nature, nevertheless keep a good deal of hidden regularity revealed in terms of string solutions to Bethe equations [3]. Very little is known on the spectra of generic type; they seem to be completely irregular. In this paper we discuss just the opposite case of the utmost regular spectra in the sense that the number of bands is finite although T/η is irrational. Moreover, the number of bands does not really depend on this ratio, being determined by another (integer) parameter. The operators with this type of spectra are true difference analogues of the finite-gap operators.

In [4], the following difference analogue of the Lamé operator (1.1) was proposed:

$$L=\frac{\theta_1(x-\ell\eta)}{\theta_1(x)}\,e^{\eta\partial_x}+\frac{\theta_1(x+\ell\eta)}{\theta_1(x)}\,e^{-\eta\partial_x}\,. \tag{1.3}$$

Here $\theta_1(x)\equiv\theta_1(x|\tau)$ is the odd Jacobi θ-function, ℓ is a non-negative integer. The coefficients are periodic functions with period 1. This operator can be made self-adjoint by the

similarity transformation $L \to g^{-1}(x)Lg(x)$ with a function $g(x)$ such that $g(x+1) = g(x)$, so the spectrum is real. The operator (1.3) first appeared in a completely different context of representations of the Sklyanin algebra as early as in 1983 [5]. Namely, L coincides with one of the four generators of the Sklyanin algebra in the functional realization found by Sklyanin. Remarkably, the operator L for positive integer values of ℓ and *arbitrary* generic η has $2\ell+1$ stable bands (and 2ℓ gaps) in the spectrum. The finite-gap property of this operator for integer ℓ was proved in [4]. It was also shown [4, 6] that the Sklyanin algebra does provide a natural algebraic framework for analyzing the spectral properties of the operator L. (A different algebraic approach to the difference analogues of the Lamé operators was proposed in [7].)

Another similarity transformation, $\tilde{L} = f^{-1}Lf$, where

$$f(x) = \prod_{j=1}^{\ell} \theta_1(x - j\eta)\,, \tag{1.4}$$

makes coefficients of the difference operator

$$\tilde{L} = e^{\eta\partial_x} + \frac{\theta_1(x+\ell\eta)\theta_1(x-(\ell+1)\eta)}{\theta_1(x)\theta_1(x-\eta)}\, e^{-\eta\partial_x} \tag{1.5}$$

double-periodic functions of x with periods 1 and τ. The limit $\eta \to 0$ gives the Lamé operator (1.1). Indeed, replacing x by $x + \frac{1}{2}\tau$ in (1.5) we obtain:

$$\tilde{L} = 2 - \eta^2(\mathcal{L} + \text{const}) + O(\eta^3)\,, \qquad \eta \to 0\,, \tag{1.6}$$

where the $\wp$-function in the $\mathcal{L}$ has periods 1 and τ (see (A3)).

Let us mention that spectral curves of the classical Lamé operator (1.1) and its Treibich-Verdier generalizations [8] for small values of ℓ were studied in [9], [10]. A detailed analysis of solutions to the difference Lamé equation at $\ell = 1$ was recently carried out in [11].

The paper is organized as follows. Sec. 2 is a continuation of the Introduction. To draw the problem in a broader context, we discuss the general notion of the finite-gap operator. In Sec. 3, a family of Bloch eigenfunctions of the operator (1.5) is constructed. These eigenfunctions are parametrized by points of the spectral curve. Sec. 4 contains equations for the edges of bands and some examples. In Sec. 5 we work out an explicit relation between the Bloch multipliers. The form of the result suggests that some hypothetical combinatorial identities for "elliptic numbers" may be relevant. At last, Sec. 6 contains some remarks on the isospectral deformations of the difference Lamé operator. In this case the coefficient in (1.5) has more poles. The location of the poles, however, is not arbitrary: they are constrained by locus equations.

2 A general view of finite-gap operators

The key idea of the modern approach to spectra of differential or difference operators is to regard the solutions $\Psi(x, E)$ to the spectral problem (say, (1.2)) as functions of E for any complex values of E and to study their analytic properties in E. In so doing it is not necessary to assume that the problem is self-adjoint, so the parameter η and the coefficients may be complex numbers.

In practice, one may try to construct a family of eigenfunctions $\Psi = \Psi(x, E, p_1, p_2, \ldots)$ depending, apart from E, on a finite number of additional parameters p_i. For instance, one of these could be the Bloch momentum k: $\Psi(x) = e^{ikx}\chi(x, E, k)$, where $\chi(x + T, E, k) = \chi(x, E, k)$. Suppose such a family does exist. Then the spectral parameters appear to be constrained by some relations $F_i(E, p_1, p_2, \ldots) = 0$, so that only one of the parameters is independent. These relations define a complex curve (a Riemann surface) in the parameter space called *the spectral curve*. The true spectral parameter is a point of the curve. This is the proper mathematical formulation of the dispersion law $E = E(k)$. Usually, this function is multi-valued. It becomes single-valued on the spectral curve (when the latter is well-defined). Moreover, the solution $\Psi(x, E)$ of the spectral problem also becomes a single-valued function on the spectral curve.

In the case of the second-order difference operators the spectral problem (1.2) has no more than two linearly independent solutions. In other words, the function E can take any of its values at most twice. The existence of such a function implies that the spectral curve is a hyperelliptic curve. Any hyperelliptic curve can be represented in the form a two-sheet covering of the complex plane of the variable E:

$$w^2 = \prod_i (E - E_i)\,, \tag{2.1}$$

where E_i are called *branch points*. For $E = E_i$ the equation (1.2) has only one linearly independent solution. The curve (2.1) is well-defined if the set of branch points is finite. Then the curve is algebraic and has finite genus. Equivalently, this means that there exists a difference operator W such that W can not be represented as a polynomial function of the difference operator in the right hand side of (1.2), and that commutes with this operator. In this case they have a set of common eigenfunctions. Equation (2.1) is then lifted to the operator relation. The parameter w is the eigenvalue of the operator W on the common eigenfunction $\Psi(x, E)$: $W\Psi(x, E) = w\Psi(x, E)$.

For self-adjoint spectral problems the branch points E_i are real numbers $E_1 < E_2 < E_3 < \ldots$. In the stable bands the Bloch momentum takes real values. The stable bands are segments of the real line $[E_{2i+1},\, E_{2i+2}]$, $i = 0, 1, \ldots$, so the branch points are just edges of bands. At the edges of bands, the Bloch solutions $\Psi(x, E_i)$ are periodic or anti-periodic.

The notion of the spectral curve is really useful in the exceptional case of the spectra of regular type mentioned in the Introduction. Recall that in the typical case the set of branch points may be even uncountable, so the spectral curve, defined as above, does not have sense. The very existence of a well-defined spectral curve of finite genus is the precise characterization of the finite-gap operators.

Then it is natural to address the inverse problem: given a hyperelliptic curve of finite genus regarded as a spectral curve of some difference operator, to find coefficients of this operator, i.e., the functions $a(x)$, $b(x)$ in (1.2). In this way, one is able to construct a representative family of finite-gap operators [12, 13]. For difference operators this was done in [14, 15, 16]. The coefficients of the operators can be expressed through Riemann's theta-functions associated with the curve.

The curve itself does not determine the operator uniquely. There is a remaining finite-parametric freedom which can be fixed by some additional data on the curve (essentially, a number of marked points). In other words, any finite-gap operator admits a class of

isospectral deformations. The coefficients of the operator, with respect to the isospectral flows, obey certain non-linear integrable equations.

Looking for formulas more effective than a bunch of Riemann's theta-fucntions, one may inquire whether they can be expressed through simpler functions, for instance elliptic ones. For a particular class of curves, namely, for special coverings of elliptic curves, this is indeed possible (see e.g. [17]). The Riemann theta-function associated with such a curve factorizes into a product of Jacobi θ-functions, so the coefficients of the operator become elliptic functions. Moreover, all the family of isospectral deformations of the operator enjoys the same property.

An important example of this phenomenon in the differential set-up is provided by the Lamé operator (1.1) for $\ell \in \mathbf{Z}_+$ and its isospectral deformations $-\partial_x^2 + u(x)$ with

$$u(x) = 2 \sum_{j=1}^{\ell(\ell+1)/2} \wp(x - x_j) + \text{ const}\,. \tag{2.2}$$

The Lamé operator itself corresponds to a very degenerate configuration when all the poles sit in one and the same point. The isospectral flows are the flows of the KdV hierarchy [12] for the potential $u(x)$. Solving say the KdV equation $\dot{u} = 6uu' - u'''$ for $u = u(x,t)$ with the initial condition $u(x,0) = \ell(\ell+1)\wp(x - x_0)$, we get a family of Schrödinger operators with elliptic potential which have the same spectral curve as the Lamé operator. The poles x_j (and the constant term) in (2.2) become t-dependent. By a direct substitution to the KdV equation, it can be shown [18] that they are constrained by the conditions

$$\sum_{j=1,\neq i}^{\ell(\ell+1)/2} \wp'(x_i - x_j) = 0\,, \qquad i = 1, 2, \ldots, \ell(\ell+1)/2\,, \tag{2.3}$$

and obey the differential equations

$$\dot{x}_j = -12 \sum_{k=1,\neq j}^{\ell(\ell+1)/2} \wp(x_j - x_k)\,. \tag{2.4}$$

Eqs. (2.3) are the famous equations defining the equilibrium locus of the elliptic Calogero-Moser system of particles. From the general theory which connects the pole dynamics of elliptic solutions of non-linear integrable equations with systems of Calogero-Moser type [18, 19] it follows that the connected component of the locus is parametrized by the Jacobian of the spectral curve of the Lamé operator. So, it is an ℓ-dimensional submanifold, spanned by higher Calogero-Moser flows, in the $\frac{1}{2}\ell(\ell+1)$-dimensional configuration space with coordinates x_j.

In Sec. 6, we present analogues of equations (2.2), (2.3) and (2.4) in the difference set-up. The isospectral flows are connected with eliptic solutions to the Volterra hierarchy.

3 Double-Bloch eigenfunctions of the difference Lamé operator and the spectral curve

In this section we study Bloch eigenfunctions of the difference Lamé operator. Following the general scheme outlined at the beginning of Sec. 2, we construct a family of eigenfunc-

tions depending on E and two additional spectral parameters. All the three parameters are constrained by two equations which define the spectral curve.

Consider the eigenvalue equation for the operator $\tilde{L}$ (1.5):

$$\psi(x+\eta) + \frac{\theta_1(x+\ell\eta)\theta_1(x-(\ell+1)\eta)}{\theta_1(x)\theta_1(x-\eta)}\,\psi(x-\eta) = E\psi(x)\,. \tag{3.1}$$

The coefficient function is double-periodic. Therefore, it is natural to look for solutions in the class of *double-Bloch functions* [4], i.e., such that $\psi(x+1) = B_1\psi(x)$, $\psi(x+\tau) = B_\tau\psi(x)$ with some constants B_1, B_τ. These are going to be the additional parameters p_i from the general scheme of Sec. 2.

Consider the function

$$\Phi(x,\zeta) = \frac{\theta_1(\zeta+x)}{\theta_1(x)\theta_1(\zeta)}\,. \tag{3.2}$$

Its monodromy properties in x are $\Phi(x+1,\zeta) = \Phi(x,\zeta)$, $\Phi(x+\tau,\zeta) = e^{-2\pi i\zeta}\Phi(x,\zeta)$, i.e., it is a double-Bloch function. Moreover, it is the simplest non-trivial (i.e., different from the exponential function) double-Bloch function since it has only one pole. This function serves as a building block for more general double-Bloch functions.

Let ℓ be a positive integer. We employ the following double-Bloch ansatz for the ψ:

$$\psi(x) = K^{x/\eta}\sum_{j=1}^{\ell} s_j(\zeta,K,E)\Phi(x-j\eta,\zeta)\,, \tag{3.3}$$

where ζ, K parametrize the Bloch multipliers of the function $\psi(x)$: $B_1 = K^{\frac{1}{\eta}}$, $B_\tau = K^{\frac{\tau}{\eta}}e^{-2\pi i\zeta}$. The coefficients s_j depend on the indicated parameters only.

Substituting (3.3) into (3.1) and computing the residues at the points $x = j\eta$, $j = 0,\ldots,\ell$, we get $\ell+1$ linear equations

$$\sum_{j=1}^{\ell} M_{ij}s_j = 0, \qquad i = 0,1,\ldots,\ell\,, \tag{3.4}$$

for ℓ unknowns s_j. Matrix elements M_{ij} of this system are given by the formula

$$\begin{aligned} M_{ij} \;&=\; K\delta_{i,j-1} - E\delta_{i,j} + K^{-1}\frac{\theta_1((j+\ell+1)\eta)\theta_1((j-\ell)\eta)}{\theta_1((j+1)\eta)\theta_1(j\eta)}\delta_{i,j+1} + \\ &+\; K^{-1}\frac{\theta_1(\zeta-(j-i+1)\eta)}{\theta_1(\zeta)}\,\frac{\theta_1((i+\ell)\eta)\theta_1((i-\ell-1)\eta)}{\theta_1(\eta)\theta_1((j-i+1)\eta)}(\delta_{i,0}-\delta_{i,1})\,. \end{aligned} \tag{3.5}$$

Here $i = 0,1,\ldots,\ell$, $j = 1,2,\ldots,\ell$. The overdetermined system (3.4) has nontrivial solutions if and only if rank of the rectangular matrix M_{ij} is less than ℓ. Let $M^{(0)}$ and $M^{(1)}$ be $\ell\times\ell$ matrices obtained from M by deleting the rows with $i = 0$ and $i = 1$, respectively. Then the values of three parameters ζ, K, E for which eq. (3.4) has solutions of the form (3.3) are determined by the system of two equations: $\det M^{(0)} = \det M^{(1)} = 0$. They indeed define a curve.

To obtain the explicit form of these equations, we expand the determinants with respect to the first row. This yields an explicit characterization of the spectral curve summarized

in the theorem below. Hereafter, the "elliptic factorial" and "elliptic binomial" notation is convenient:

$$
\begin{aligned}
&[n]! = \prod_{j=1}^{n} [j]\,, \qquad [j] \equiv \theta_1(j\eta)/\theta_1(\eta)\,, \\
&\left[\begin{array}{c} n \\ m \end{array} \right] \equiv \frac{[n]!}{[m]![n-m]!}\,.
\end{aligned}
\tag{3.6}
$$

Theorem 3.1 *The difference Lamé equation (3.1) has double-Bloch solutions of the form (3.3) if and only if the spectral parameters ζ, K, E obey the equations*

$$
\begin{aligned}
&\sum_{j=0}^{\ell} (-1)^j K^{-j} \theta_1(\zeta - j\eta) \left[\begin{array}{c} \ell \\ j \end{array} \right] A_j^{(\ell)}(E) = 0\,, \\
&\sum_{j=0}^{\ell+1} (-1)^j K^{-j} \theta_1(\zeta - j\eta)[j-1] \left[\begin{array}{c} \ell+1 \\ j \end{array} \right] A_{|j-1|}^{(\ell)}(E) = 0\,,
\end{aligned}
\tag{3.7}
$$

where $A_j^{(\ell)}(E)$ are polynomials of $(\ell - j)$-th degree explicitly given by the determinant formula

$$
A_{\ell-s}^{(\ell)}(E) = \left[\begin{array}{c} \ell \\ s \end{array} \right] \left[\begin{array}{c} 2\ell \\ s \end{array} \right]^{-1} \det \left(E\delta_{i,j} + \frac{[-i]}{[\ell+1-i]}\delta_{i,j-1} + \frac{[2\ell+2-i]}{[\ell+1-i]}\delta_{i,j+1} \right)_{1\le i,j\le s}
\tag{3.8}
$$

(here $0 \le s \le \ell$), $A_\ell^{(\ell)} = 1$.

Let us list some useful properties of the polynomials $A_j^{(\ell)}(E)$. First, they obey the recurrence relation:

$$
A_{\ell-s-1}^{(\ell)}(E) = \frac{[\ell-s]}{[2\ell-s]} E A_{\ell-s}^{(\ell)}(E) + \frac{[s]}{[2\ell-s]} A_{\ell-s+1}^{(\ell)}(E)
\tag{3.9}
$$

with the initial condition $A_\ell^{(\ell)}(E) = 1$, $A_{\ell-1}^{(\ell)}(E) = ([\ell]/[2\ell])E$. Next, it is clear from (3.9) that

$$
A_{\ell-s}^{(\ell)}(-E) = (-1)^s A_{\ell-s}^{(\ell)}(E)\,, \qquad 0 \le s \le \ell\,.
\tag{3.10}
$$

The equations (3.7) define a Riemann surface $\tilde{\Gamma}$, which covers the complex plane. The monodromy properties of the θ-function (see (A4) in the Appendix) make it clear that this surface is invariant under the transformation

$$
\zeta \longmapsto \zeta + \tau\,, \qquad K \longmapsto K e^{2\pi i \eta}\,.
\tag{3.11}
$$

The factor of the $\tilde{\Gamma}$ over this transformation is an algebraic curve Γ, which is a ramified covering of the elliptic curve with periods $1, \tau$. It is clear from (3.7), (3.10) that the curve admits the involution

$$
(\zeta, K, E) \longmapsto (\zeta, -K, -E)\,,
\tag{3.12}
$$

so the spectrum is symmetric with respect to the reflection $E \to -E$. Another result of [4], which is not so easy to see from (3.7), is that the curve Γ is at the same time a hyperelliptic curve.

Theorem 3.2 *The curve Γ is a hyperelliptic curve of genus $g = 2\ell$. The hyperelliptic involution is given by*

$$(\zeta,\, K,\, E) \longmapsto (2N\eta - \zeta,\, K^{-1}, E)\,, \qquad N = \frac{1}{2}\ell(\ell+1)\,. \tag{3.13}$$

The points $P = (\zeta, K, E) \in \Gamma$ of the curve parametrize double-Bloch solutions $\psi(x) = \psi(x, P)$ to eq. (3.1), and the solution $\psi(x, P)$ corresponding to each point $P \in \Gamma$ is unique up to a constant multiplier.

For the proof see [4]. Here we give a few remarks. The involution (3.13) looks best in terms of the function $\Psi(x) = \psi(x)\prod_{j=1}^{\ell}\theta_1(x - j\eta)$ which satisfies the eigenvalue equation $L\Psi = E\Psi$ with the L as in (1.3) (cf. (1.4)). Then the hyperelliptic involution simply takes $\Psi(x)$ to $\Psi(-x)$. The genus of the curve can be found by counting the number of the fixed points of this involution. At the fixed points the two solutions, $\Psi(x)$ and $\Psi(-x)$, are linearly dependent: $\Psi(-x) = r\Psi(x)$. Writing out the eigenvalue equation at $x = 0$, we obtain the necessary condition $\Psi(-\eta) = \Psi(\eta)$, so $r = 1$. The ansatz (3.3) for ψ is equivalent to the ansatz

$$\Psi(x) = K^{x/\eta}\prod_{j=1}^{\ell}\theta_1(x + y_j)$$

with $\sum_j y_j = \zeta$. At $K = 1$, the dimension of the linear space of even functions of this form is known to be $2\ell + 1$. Adding images of the fixed points under the involution (3.12), we eventually get $4\ell + 2$ fixed points, so, by the Riemann-Hurwitz formula, the genus is equal to 2ℓ.

Taking into account the symmetry $E \to -E$, we can write the equation of the hyperelliptic curve in the standard form:

$$w^2 = \prod_{i=1}^{2\ell+1}(E^2 - E_i^2)\,. \tag{3.14}$$

The hyperelliptic involution takes (w, E) to $(-w, E)$.

In (3.14), w is the eigenvalue of a non-trivial operator W commuting with L on their common eigenfunction. The explicit form of the operator W was found in [20]:

$$W = \varphi_\ell(x)\sum_{k=0}^{2\ell+1}(-1)^k \left[\begin{array}{c} 2\ell+1 \\ k \end{array}\right] \frac{\theta_1(x + (2\ell - 2k + 1)\eta)}{\prod_{j=0}^{2\ell-k+1}\theta_1(x + j\eta)\prod_{j'=1}^{k}\theta_1(x - j'\eta)} e^{(2\ell-2k+1)\eta\partial_x}\,,$$

where $\varphi_\ell(x) = \prod_{j=0}^{2\ell}\theta_1(x + (j-\ell)\eta)$.

Let us conclude this section by examining the behaviour of the spectral curve in the vicinity of its "infinite points", i.e., the points at which the function E has poles. From (3.7) we conclude that there are two such points: $\infty_+ = (\zeta \to 0,\, K \to \infty,\, E \to \infty)$ and $\infty_- = (\zeta \to 2N\eta,\, K \to 0,\, E \to \infty)$. In the neighbourhood of $\infty_\pm$ $E = K^{\pm 1} + o(K^{\pm 1})$. In terms of the variables (w, E) these points are $\infty_\pm = (w \to \pm\infty,\, E \to \infty)$.

4 Edges of bands

The edges of bands $\pm E_i$, i.e., the branch points of the two-sheet covering (3.14), are values of the function $E = E(P)$ at the fixed points of the hyperelliptic involution. As is clear

from (3.13), the fixed points lie above the points $\zeta = N\eta + \omega_a$, where ω_a are the half-periods: $\omega_1 = 0$, $\omega_2 = \frac{1}{2}$, $\omega_3 = \frac{1}{2}(1+\tau)$, $\omega_4 = \frac{1}{2}\tau$. The corresponding values of K are determined from (3.11). Then the set of the branch points E_i is fixed by Theorem 3.1.

Corollary 4.1 *Let $\mathcal{E}_a$, $a = 1, \ldots, 4$ be the set of common roots of the polynomial equations*

$$
\begin{aligned}
&\sum_{j=0}^{\ell} (-1)^j \theta_a((N-j)\eta) \left[\begin{array}{c} \ell \\ j \end{array} \right] A_j^{(\ell)}(E) = 0 \,, \\
&\sum_{j=0}^{\ell+1} (-1)^j \theta_a((N-j)\eta)[j-1] \left[\begin{array}{c} \ell+1 \\ j \end{array} \right] A_{|j-1|}^{(\ell)}(E) = 0 \,,
\end{aligned}
\tag{4.1}
$$

where θ_a are Jacobi θ-functions[1]*. Then the set of the edges of bands $\pm E_i$ is the union of $\bigcup_{a=1}^4 \mathcal{E}_a$ and its image under the reflection $E \to -E$.*

Let us give two examples. At $\ell = 1$, the set $\mathcal{E}_1$ is empty while $\mathcal{E}_a$ for $a = 2, 3, 4$ contains one point. From (4.1) we find:

$$
E_\alpha = 2\, \frac{\theta_{\beta+1}(\eta)\theta_{\gamma+1}(\eta)}{\theta_{\beta+1}(0)\theta_{\gamma+1}(0)} \,,
$$

where $\{\alpha,\, \beta,\, \gamma\}$ is any cyclic permutation of $\{1,\, 2,\, 3\}$. At $\ell = 2$, the set $\mathcal{E}_1$ has two elements E_1, E_2 obtained as solutions of the quadratic equation $[2]E^2 + [2]^3 E + 2[4] = 0$, so that

$$
E_{1,2} = \frac{1}{2} \left(\frac{\theta_1^2(2\eta)}{\theta_1^2(\eta)} \pm \sqrt{ \frac{\theta_1^4(2\eta)}{\theta_1^4(\eta)} - 8 \frac{\theta_1(4\eta)}{\theta_1(2\eta)} } \right) .
$$

For $a = 2, 3, 4$ each set $\mathcal{E}_a$ has one element E_{a+1}:

$$
E_{a+1} = \frac{\theta_1(2\eta)\theta_a(2\eta)}{\theta_1(\eta)\theta_a(\eta)} \,.
$$

In general, it is possible to prove [4] that

$$
\#(\mathcal{E}_1) = \left\{ \begin{array}{ll} \frac{1}{2}(\ell-1)\,, & \ell \ \text{odd} \\ \\ \frac{1}{2}\ell+1\,, & \ell \ \text{even} \end{array} \right. \qquad \#(\mathcal{E}_{2,3,4}) = \left\{ \begin{array}{ll} \frac{1}{2}(\ell+1)\,, & \ell \ \text{odd} \\ \\ \frac{1}{2}\,\ell\,, & \ell \ \text{even}\,. \end{array} \right. \tag{4.2}
$$

Note that $\#\Big(\cup_{a=1}^4 \mathcal{E}_a\Big) = 2\ell + 1$ that agrees with (3.14).

5 Relation between the Bloch multipliers

To simplify equations of the curve (3.7), one can try to eliminate one of the variables and obtain a single equation for the other two. Here we show how to eliminate E. This leads

[1]Definitions and transformation properties of the Jacobi θ-functions $\theta_a(x|\tau)$, $a = 1, 2, 3, 4$, are listed in the Appendix. For brevity, we write $\theta_a(x|\tau) \equiv \theta_a(x)$.

to a closed relation between the two Bloch multipliers of the function (3.3) (parametrized through ζ and K). Its form (see (5.6), (5.8) below) suggests an interpretation in terms of hypothetical combinatorial identities for elliptic numbers.

At the first glance, the elimination of E from eqs. (3.7) is hardly possible at all. Nevertheless, there is an alternative argument leading directly to the relation between the Bloch multipliers. Here it is more convenient to deal with the difference Lamé operator in the gauge equivalent form (1.3). Our construction is based on the following simple lemma [6].

Lemma 5.1 *Let $\Psi(x)$ be any solution to the equation*

$$\frac{\theta_1(x-\ell\eta)}{\theta_1(x)}\Psi(x+\eta)+\frac{\theta_1(x+\ell\eta)}{\theta_1(x)}\Psi(x-\eta)=E\Psi(x) \tag{5.1}$$

in the class of entire functions on the complex plane of the variable x, then

$$\Psi(j\eta)=\Psi(-j\eta)\,, \qquad j=1,2,\ldots,\ell\,. \tag{5.2}$$

This assertion follows from the specific location of zeros and poles of the coefficients of eq. (5.1). Indeed, putting $x=0$ in (5.1), we have $\Psi(\eta)=\Psi(-\eta)$. The proof can be completed by induction. At $x=\pm\ell\eta$ one of the coefficients in the l.h.s. of (5.1) vanishes, so the chain of relations (5.2) truncates at $j=\ell$.

Remarkably, the conditions (5.2) and the ansatz

$$\Psi(x)=K^{x/\eta}\left(\prod_{j=1}^{\ell}\theta_1(x-j\eta)\right)\sum_{m=1}^{\ell}s_m(K,\zeta)\Phi(x-m\eta,\,\zeta) \tag{5.3}$$

for Ψ (equivalent to the ansatz (3.3) for ψ) with the same function $\Phi(x,z)$ given by (3.2) allow one to find the relation between the Bloch multipliers even without explicit use of the difference Lamé equation (5.1). Plugging (5.3) into (5.2), we obtain ℓ equalities (for $m=1,2,\ldots,\ell$):

$$K^m s_m=(-1)^{\ell}K^{-m}\theta_1(2m\eta)\left(\prod_{j=1,\neq m}^{\ell}\frac{\theta_1((m+j)\eta)}{\theta_1((m-j)\eta)}\right)\sum_{n=1}^{\ell}\Phi\Big(-(m+n)\eta,\,\zeta\Big)\,s_n\,. \tag{5.4}$$

This is a system of linear homogeneous equations for s_n. It has nontrivial solutions if and only if its determinant is equal to zero, whence we obtain the equation connecting ζ and K:

$$\det\Big(K^{2m}\delta_{mn}+G_{mn}(\zeta)\Big)_{1\le m,n\le\ell}=0\,, \tag{5.5}$$

where

$$G_{mn}(\zeta)=(-1)^{\ell+1}[2m]\left(\prod_{j=1,\neq m}^{\ell}\frac{[m+j]}{[m-j]}\right)\Phi(-(m+n)\eta,\,\zeta)\,.$$

This equation defines a curve, which is the image of the spectral curve Γ under the projection that takes (ζ,K,E) to (ζ,K).

The equation of the spectral curve (5.5) can be represented in the form

$$\sum_{j=0}^{N}(-1)^j C_j^{(\ell)}(\eta)\theta_1(\zeta-2j\eta)K^{2(N-j)}=0\,, \tag{5.6}$$

where $N = \frac{1}{2}\ell(\ell+1)$ and $C_j^{(\ell)}(\eta)$ are some coefficients depending only on η and τ such that $C_j^{(\ell)}(\eta) = C_{N-j}^{(\ell)}(\eta)$, $C_0^{(\ell)}(\eta) = 1$. To see this, we expand the determinant (5.5) in powers of K with the help of the identity

$$\det\left(\frac{\theta_1(x_i + x_j + \zeta)}{\theta_1(x_i + x_j)}\right)_{1\le i,j\le n} = \frac{\theta_1^{n-1}(\zeta)\theta_1(\zeta + 2\sum_{i=1}^n x_i)}{\prod_{i=1}^n \theta_1(2x_i)} \prod_{i<j}^n \frac{\theta_1^2(x_i - x_j)}{\theta_1^2(x_i + x_j)}$$

(a particular case of the formula for the elliptic Cauchy determinant). Let Λ be the set $\{1, 2, \ldots, \ell\}$. For any subset $J \subseteq \Lambda$, $\Lambda \backslash J$ is its complement, and $\sigma(J) = \sum_{j\in J} j$. Evaluating the elliptic Cauchy determinant at $x_n = -2\eta n$, we get:

$$\begin{aligned} &\det\Big(K^{2m}\delta_{mn} + G_{mn}(\zeta)\Big)_{1\le m,n\le\ell} \\ &= \sum_{J\subseteq\Lambda} \frac{\theta_1(\zeta - 2\sigma(J)\eta)}{\theta_1(\zeta)} K^{2N-2\sigma(J)}(-1)^{\sigma(J)} \prod_{k\in J}\prod_{k'\in\Lambda\backslash J} \frac{[k+k']}{[|k-k'|]}\,. \end{aligned} \tag{5.7}$$

Thus, the coefficient $C_j^{(\ell)}$ reads

$$C_j^{(\ell)} = \sum_{J\subseteq\Lambda,\sigma(J)=j} \prod_{k\in J}\prod_{k'\in\Lambda\backslash J} \frac{[k+k']}{[|k-k'|]}\,. \tag{5.8}$$

The symmetry $j \leftrightarrow N - j$ is now transparent. Note that the sum in (5.8) runs over partitions of the number j into *distinct* parts not exceeding ℓ.

We note that for any elliptic module τ

$$\lim_{\eta\to 0} C_j^{(\ell)} = \left(\begin{array}{c} N \\ j \end{array}\right) \tag{5.9}$$

(the usual binomial coefficient). To see this, consider the limiting case $\tau \to i\infty$. We have: $\exp(-\pi i\tau/4)\theta_1(x|\tau) \to 2\sin(\pi x)$ as $\tau \to i\infty$ (see (A2)), so

$$[j] \longrightarrow \frac{\sin(\pi\eta j)}{\sin(\pi\eta)} \equiv (j)_q\,, \qquad q = e^{2\pi i\eta}\,.$$

Then (5.9) follows from the identity [21]

$$\sum_{J\subseteq\Lambda} z^{\sigma(J)} \prod_{k\in J}\prod_{k'\in\Lambda\backslash J} \frac{(k+k')_q}{(|k-k'|)_q} = \prod_{1\le j\le k\le\ell} (1 + zq^{j+k-\ell-1}) \tag{5.10}$$

which is a specialization of the C_ℓ-type Weyl denominator formula[2]. A detailed combinatorial analysis of the limiting cases $\tau \to i\infty$ or $\tau \to 0$ of the difference Lamé operators can be found in [21].

[2] I am grateful to A.N.Kirillov for pointing out this identity and drawing my attention to the paper [21].

6 Isospectral deformations of the difference Lamé operator and locus equations

Finally, let us comment on isospectral deformations of the difference Lamé operator. We are going to present difference analogues of the operator (2.2) and of the locus equations (2.3).

Instead of (3.1), consider the equation

$$\psi(x+\eta) + c(x)\psi(x-\eta) = E\psi(x) \tag{6.1}$$

with a more general coefficient $c(x)$, which is an elliptic function represented in the form

$$c(x) = \frac{\rho(x+\eta)\rho(x-2\eta)}{\rho(x)\rho(x-\eta)}, \qquad \rho(x) = \prod_{j=1}^{\ell(\ell+1)/2} \theta_1(x-x_j)\,. \tag{6.2}$$

Note that in the case of the difference Lamé operator the configuration of zeros of the $\rho(x)$ is very specific:

$$\rho(x) = \prod_{1\le j\le k\le \ell} \theta_1(x+(j+k-\ell-1)\eta)\,, \tag{6.3}$$

so all but two cancel in the $c(x)$.

The isospectral flows are the flows of the Volterra hierarchy for the $c(x)$. The first equation of the hierarchy,

$$\frac{\partial c(x)}{\partial t} = -c(x)(c(x+\eta)-c(x-\eta))\,, \tag{6.4}$$

is the compatibility condition of the spectral problem (6.1) and the linear problem

$$\frac{\partial \psi(x)}{\partial t} = c(x)c(x-\eta)\psi(x-2\eta)\,.$$

Recall that changing the variables as $t \to \frac{1}{3}\eta^3 t$, $x \to x-2\eta t$ and setting $c(x) = 1-\eta^2 u(x)$, one gets the KdV equation for u as $\eta \to 0$.

Substituting the pole ansatz (6.2) into the Volterra equation and requiring the residues at the poles to be zero, we get the following two systems of equations ($j = 1, 2, \ldots, \frac{1}{2}\ell(\ell+1)$):

$$\begin{aligned} \frac{\theta_1'(0)}{\theta_1(2\eta)}\,\dot{x}_j &= \prod_{k=1,\neq j}^{\ell(\ell+1)/2} \frac{\theta_1(x_j-x_k+2\eta)\theta_1(x_j-x_k-\eta)}{\theta_1(x_j-x_k+\eta)\theta_1(x_j-x_k)}\,, \\ \frac{\theta_1'(0)}{\theta_1(2\eta)}\,\dot{x}_j &= \prod_{k=1,\neq j}^{\ell(\ell+1)/2} \frac{\theta_1(x_j-x_k-2\eta)\theta_1(x_j-x_k+\eta)}{\theta_1(x_j-x_k-\eta)\theta_1(x_j-x_k)}\,, \end{aligned} \tag{6.5}$$

where $\dot{x}_j = \partial_t x_j$. Solving these equations, or, equivalently, the Volterra equation with the initial condition (6.3), one arrives at a family of isospectral deformations of the difference Lamé operator. The two systems (6.5) must be satisfied simultaneously, therefore, the right hand sides are identical. Whence we obtain the necessary conditions for solutions to exist:

$$\prod_{k=1,\neq j}^{\ell(\ell+1)/2} \frac{\theta_1(x_j-x_k+2\eta)\theta_1^2(x_j-x_k-\eta)}{\theta_1(x_j-x_k-2\eta)\theta_1^2(x_j-x_k+\eta)} = 1 \tag{6.6}$$

that is the difference analogues of (2.3). The results of [4] imply that these equations define an equilibrium locus of the Ruijsenaars-Schneider model. Like in the differential case, the locus is not compact. Its closure contains, in particular, the point corresponding to the degenerate configuration (6.3). From the general arguments of ref. [4] it follows that the connected component of the locus that contains this point at the boundary is ℓ-dimensional. Expanding (6.6) in $\eta \to 0$, we do get (2.3).

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Appendix. Theta-functions

We use the following definition of the Jacobi θ-functions:

$$
\begin{aligned}
\theta_1(x|\tau) &= -\sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i(x+\frac{1}{2})(k+\frac{1}{2})\right),\\
\theta_2(x|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i x(k+\frac{1}{2})\right),\\
\theta_3(x|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi i x k\right),\\
\theta_4(x|\tau) &= \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi i(x+\frac{1}{2})k\right).
\end{aligned}
\tag{A1}
$$

They can be represented as infinite products:

$$
\begin{aligned}
\theta_1(x|\tau) &= 2\sin(\pi x)e^{\pi i\tau/4}\prod_{k=1}^{\infty}\left(1-e^{2\pi ik\tau}\right)\left(1-e^{2\pi i(k\tau+x)}\right)\left(1-e^{2\pi i(k\tau-x)}\right),\\
\theta_2(x|\tau) &= 2\cos(\pi x)e^{\pi i\tau/4}\prod_{k=1}^{\infty}\left(1-e^{2\pi ik\tau}\right)\left(1+e^{2\pi i(k\tau+x)}\right)\left(1+e^{2\pi i(k\tau-x)}\right),\\
\theta_3(x|\tau) &= \prod_{k=1}^{\infty}\left(1-e^{2\pi ik\tau}\right)\left(1+e^{\pi i(2k\tau-\tau+2x)}\right)\left(1+e^{\pi i(2k\tau-\tau-2x)}\right),\\
\theta_4(x|\tau) &= \prod_{k=1}^{\infty}\left(1-e^{2\pi ik\tau}\right)\left(1-e^{\pi i(2k\tau-\tau+2x)}\right)\left(1-e^{\pi i(2k\tau-\tau-2x)}\right).
\end{aligned}
\tag{A2}
$$

The $\wp$-function is related to the θ_1 as follows:

$$\wp(x|1/2, \tau/2) = -\frac{d^2}{dx^2}\log\theta_1(x|\tau) + \text{const}\,. \tag{A3}$$

Throughout the paper we write $\theta_a(x|\tau) = \theta_a(x)$.

The transformation properties of the theta-functions for shifts by (half) periods are:

$$\theta_a(x \pm 1) = (-1)^{\delta_{a,1}+\delta_{a,2}}\theta_a(x)\,, \qquad \theta_a(x \pm \tau) = (-1)^{\delta_{a,1}+\delta_{a,4}}e^{-\pi i\tau \mp 2\pi i x}\theta_a(x)\,, \tag{A4}$$

$$\begin{aligned}
&\theta_1(x \pm \tfrac{1}{2}) = \pm\theta_2(x)\,, \\
&\theta_1(x \pm \tfrac{\tau}{2}) = \pm i e^{-\frac{1}{4}\pi i\tau \mp \pi i x}\theta_4(x)\,, \\
&\theta_1(x \pm \tfrac{1+\tau}{2}) = \pm e^{-\frac{1}{4}\pi i\tau \mp \pi i x}\theta_3(x)\,.
\end{aligned} \tag{A5}$$

References

[1] E.L.Ince, *Further investigations into the periodic Lamé functions*, Proc. Roy. Soc. Edinburgh **60** (1940) 83-99

[2] M.Azbel, *The energy spectrum of conducting electron in magnetic field*, Zh. Eksp Teor. Fiz. **46** (1964) 929-946; D.R.Hofstadter, *Energy levels and wave functions for Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. **B14** (1976) 2239-2249

[3] A.Abanov, J.Talstra and P.Wiegmann, *Hierarchical structure of Azbel-Hofstadter problem: Strings and loose ends of Bethe ansatz*, Nucl. Phys. **B525** (1998) 571-596

[4] I.Krichever and A.Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra*, Usp. Mat. Nauk, **50:6** (1995) 3-56, hep-th/9505039

[5] E.K.Sklyanin, *On some algebraic structures related to the Yang-Baxter equation*, Funk. Anal. i ego Pril. **16:4** (1982) 27-34; *On some algebraic structures related to the Yang-Baxter equation. Representations of the quantum algebra*, Funk. Anal i ego Pril. **17:4** (1983) 34-48

[6] A.Zabrodin, *On the spectral curve of the difference Lamé operator*, Int. Math. Research Notices, No. 11 (1999) 589-614

[7] G.Felder and A.Varchenko, *Algebraic Bethe ansatz for the elliptic quantum group* $E_{\tau,\eta}(sl_2)$, Nucl. Phys. **B480** (1996) 485-503

[8] A.Treibich and J.-L.Verdier, *Solitons elliptiques*, Grothendieck Festschrift, ed.: P.Cartier et al, Progress in Math. **88**, Birkhäuser, Boston, 1990

[9] V.Enolskii and J. Eilbeck, *On the two-gap locus for the elliptic Calogero-Moser model*, J. Phys. A **28** (1995) 1069-1088

[10] A.O.Smirnov, *Elliptic solutions of the Korteweg-De Vries equation*, Matem. Zametki, **45:6** (1989) 66-73

[11] S.N.M.Ruijsenaars, *Relativistic Lamé functions: the special case $g = 2$*, J. Phys. A: Math. Gen. **32** (1999) 1737-1772

[12] B. Dubrovin, V. Matveev and S. Novikov, *Non-linear equations of Korteweg-de Vries type, finite zone linear operators and Abelian varieties*, Uspekhi Mat. Nauk **31:1** (1976) 55-136

[13] I.M. Krichever, *Nonlinear equations and elliptic curves*, Modern problems in mathematics, Itogi nauki i tekhniki, VINITI AN USSR **23** (1983)

[14] E.Date and S.Tanaka, *Exact solutions of the periodic Toda lattice* Prog. Theor. Phys. **5** (1976), 457-465

[15] D.Mumford, *Algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations*, Proceedings Int. Symp. Algebraic geometry, Kyoto, 1977, 115-153, Kinokuniya Book Store, Tokyo, 1978

[16] I.M.Krichever, *Algebraic curves and non-linear difference equations*, Uspekhi Mat. Nauk **33:4** (1978) 215-216

[17] E.Belokolos, A.Bobenko, V.Enolskii, A.Its and V.Matveev, Algebraic-geometrical approach to nonlinear integrable equations, Berlin: Springer, 1994

[18] H.Airault, H.McKean and J.Moser, *Rational and elliptic solutions of the KdV equation and related many-body problem*, Comm. Pure and Appl. Math. **30** (1977) 95-125

[19] I.M.Krichever, *Elliptic solutions of Kadomtsev-Petviashvilii equation and integrable systems of particles*, Func. Anal. Appl. **14:4** (1980) 282-290

[20] G.Felder and A.Varchenko, *Algebraic integrability of the two-body Ruijsenaars operator*, q-alg/9610024

[21] J.F.Van Diejen and A.N.Kirillov, *Formulas for q-spherical functions using inverse scattering theory of reflectionless Jacobi operators*, Hokkaido University preprint series in mathematics, # 430 (1998)